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Rings whose finitely generated modules are extending

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Abstract

A module M is called an extending (or CS) module, if every submodule of M is essential in a direct summand of M. In this paper we show that a ring R is right noetherian if every finitely (or 2-) generated right R-module is extending.

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This paper can be considered as a supplement to the investigation in Dung–Smith [2], where several interesting results were obtained on rings for which certain modules are extending. Among them, a complete description of right non-singular or commutative rings whose 2-generated right modules are extending, is established. However, the general case still remains open. It is even unknown, whether such a rig is right noetherian or not.

In this paper we provide an answer to this question by showing that the ring under consideration is right noetherian. Our result yields that a for a ring R, every countably generated R-module is extending, implies that every R-module is extending. In fact, we consider the question in a more general setting of modules and obtain the mentioned result as a consequence.

Note that it is easy to give an example showing that a ring R is not necessarily right noetherian if every cyclic right R-module is extending (see e.g. the example in [4, p. 214 f]).

Throughout, we consider associative rings with identity and all modules are unitary. For a module M over a ring R we write M_R to indicate that M is a right R-module. The socle and the Jacobson radical of M are denoted by $Soc(M)$ and $J(M)$, respectively. The module M is called *semisimple if* $Soc(M) = M$.

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Recall that a module M is called an *extending module* if every submodule of M is essential in a direct summand of M. For a detailed study of extending modules we refer to the book [3].

Let M be a module. Then any (finitely generated) submodule of a factor module of M is called a *(finitely generated) subfactor* of M. The next result follows easily from [9, Theorem 1]. (See also [2, Lemma 1].)

Lemma 1. *Let M be a finitely generated module such that every cyclic subfactor of M is extending. Then every factor of M has finite uniform dimension.*

A right R-module M is called a *V-module* if every simple right R-module is M-injective, or equivalently, if any submodule of M is an intersection of maximal submodules of M (see e.g. [3]).

Lemma 2 (Huynh et al. [7]). *Let M be a V-module such that every factor module of M has finite uniform dimension. Then M is noetherian.*

Lemma 3 (Osofsky [8, Lemma B]). *Let M be a uniserial module with unique composition series $M \supset U \supset V \supset 0$. Then $M \oplus (U/V)$ is not extending.*

The following result can be found in [6, Lemma 1.1].

Lemma 4. *Let M be an extending, finitely generated right R-module. If $M/Soc(M)$ has finite uniform dimension, then M has finite uniform dimension.*

For a module M_R we denote by $\sigma[M]$ the full subcategory of Mod-R whose objects are submodules of M-generated modules (cf. Wisbauer [10]).

For convenience, we say that a module M satisfies the condition (*) if every finitely generated module in $\sigma[M]$ is extending. The following is the main result:

Theorem 5. *Let M be a finitely generated right R-module satisfying (*). Then M is noetherian.*

Proof. Let M_R be a finitely generated module satisfying (*), and let $\{S_\alpha\}$ be the socle series of M defined as follows:

$$S_1 = Soc(M),\ S_\alpha/S_{\alpha-1} = Soc(M/S_{\alpha-1})$$

and for a limit ordinal α,

$$S_\alpha = \bigcup_{\beta<\alpha} S_\beta\ .$$

Put $S = \bigcup\{S_\alpha\}$. Then the module $H = M/S$ has zero socle. We first aim to show that H is noetherian. This is trivial if $H = 0$. Therefore, we assume that $H \neq 0$.

By Lemma 1, we know that every factor module of H has finite uniform dimension. Hence it is enough to show that H is a V-module, since we may then apply Lemma 2 to obtain that H is noetherian. Since H is extending, H is a direct sum of finitely many uniform modules. Hence we may assume that H is uniform without loss of generality.

Let X be an arbitrary simple right R-module. If $X \notin \sigma[H]$, then clearly X is H-injective. Hence it remains to consider the case that $X \in \sigma[H]$. Then, it is easy to see that $X \in \sigma[M]$. Moreover, $K = X \oplus H$ is finitely generated. Hence by (*), K is extending. Let V be a submodule of H and f be a homomorphism of V into X. Let $U = \{a - f(a) \mid a \in V\}$. Then U is essential in a direct summand U^* of K, say

$$K = U^* \oplus U_1$$

for some submodule U_1 of K. Since $U^* \cap X = 0$ (here we may put $X = X \oplus 0$) and since $Soc(K) = X$ we must have $X \subseteq U_1$. Moreover, $U^* \oplus X$ has uniform dimension 2. Hence X is essential in U_1. But X is closed in K, therefore $X = U_1$, i.e.

$$U^* \oplus U_1 = U^* \oplus X = X \oplus H.$$

Now if π be the projection of $U^* \oplus X$ onto X, then $\hat{f} = (\pi|H)$ is an extension of f from H to X. This shows that X is H-injective. Thus H is noetherian by Lemma 2, as desired.

Now we consider S. If $S_3 \neq 0$ (S_3 is the third socle of M), then there is a cyclic submodule Y of S_3 such that Y/S_2' is simple where $S_2' = S_2 \cap Y$. Moreover, since Y is extending and has finite uniform dimension, we have

$$Y = Y_1 \oplus \cdots \oplus Y_k,$$

where each Y_i is uniform. It is clear that one of the Y_i say Y_1, must have Loewy length 3. Again, since $Y_1/Soc(Y_1)$ is extending,

$$Y_1/Soc(Y_1) = T_1 \oplus \cdots \oplus T_m,$$

where each T_i is uniform and one of T_j, say T_1, has Loewy length 2. Now let W be the inverse image of T_1 and W_1 be the inverse image of $Soc(T_1)$ in Y_1, then W is a uniserial module with unique composition series

$$W \supset W_1 \supset Soc(Y_1).$$

By Lemma 3, $W \oplus (W_1/Soc(Y_1))$ is not an extending module. This is a contradiction to (*). Thus S has Loewy length at most 2. By Lemma 1, we easily see that S is a module of finite composition length. Since M/S is noetherian, it follows that M is noetherian, as desired. □

If we put $M = R$, then in the preceding proof, we need only to assume that any 2-generated right R-module is extending, to obtain that R is right noetherian. Moreover, following this proof, the factor ring R/S has zero right socle and it is a right noetherian, right V-ring. Hence $J(R)$ is contained in S, so $J(R)^2 = 0$. Therefore we get the following:

Corollary 6. *Let R be a ring such that every 2-generated right R-module is extending. Then R is right noetherian and $J(R)^2 = 0$. In particular, if $R/J(R)$ is von Neumann regular, then R is right artinian.*

We note that a ring R is not necessarily right artinian if every finitely generated right R-module is extending: every right and left SI-domain has this property (cf. [2, Theorem 13]), but it may not be right artinian.

The following can be obtained from Theorem 5.

Theorem 7. *For a right R-module M, the following conditions are equivalent:*

(a) *every module in $\sigma[M]$ is extending,*

(b) *every countably generated module in $\sigma[M]$ is extending.*

Moreover, we can also show, by using Theorem 5 and Corollary 6, that for a ring R the following are equivalent:

(a) the injective hull $E(R_R)$ of R_R is finitely generated and every 2-generated right R-module is extending,

(b) $R/J(R)$ is von Neumann regular and every 2-generated right R-module is extending,

(c) every (countably generated) right R-module is extending.

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